Continuous and discontinuous piecewise linear solutions of the linearly forced inviscid Burgers equation

Hans Lundmark* Jacek Szmigielski†

March 31, 2008

Abstract

We study a class of piecewise linear solutions to the inviscid Burgers equation driven by a linear forcing term. Inspired by the analogy with peakons, we think of these solutions as being made up of solitons situated at the breakpoints. We derive and solve ODEs governing the soliton dynamics, first for continuous solutions, and then for more general shock wave solutions with discontinuities. We show that triple collisions of solitons cannot take place for continuous solutions, but give an example of a triple collision in the presence of a shock.

1 Introduction

The subject of this paper is piecewise linear solutions of the PDE

$$(u_t + uu_x)_{xx} = 0, \tag{1.1}$$

which we earlier [1] have called the *derivative Burgers equation*. This name refers of course to the well-known Burgers equation $u_t + uu_x = \nu u_{xx}$ and its special case the inviscid Burgers equation $u_t + uu_x = 0$, which is the prototype equation for studying shock wave solutions of hyperbolic conservation laws. In some applications one considers also *forced* Burgers equations with terms of the form $F(x,t)$ on the right-hand side, often written as $F = -\partial V/\partial x$ with a potential V. Since equation (1.1) is equivalent to $u_t + uu_x = A(t)x + B(t)$, it is perhaps more appropriate to talk about it as a *forced inviscid Burgers equation with linear force* (or quadratic potential). Moreover, the latter equation can be rewritten as

$$u_t + \frac{1}{2}(u^2)_x = A(t)x + B(t), \tag{1.2}$$

*Department of Mathematics, Linköping University, SE-581 83 Linköping, SWEDEN; halun@mai.liu.se

†Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, Saskatchewan, S7N 5E6, CANADA; szmigiel@math.usask.ca

which makes sense for a much larger class of functions than just $u \in C^1(\mathbf{R})$. For example, if $u \in L^2_{\mathrm{loc}}(\mathbf{R})$ we can interpret (1.2) to hold in the sense of distributions. One could work with distributions acting on test functions $\psi(x,t) \in \mathcal{D}(\mathbf{R}^2)$, but the following simpler interpretation is sufficient for our purposes here: we view $u(x,t)$ as a mapping that takes a real number t to a function $u(\cdot,t) \in L^2_{\mathrm{loc}}(\mathbf{R})$ which we can identify with a distribution in $\mathcal{D}'(\mathbf{R})$. The derivative with respect to x is then the distributional derivative defined by its action on a test function $\psi(x) \in \mathcal{D}(\mathbf{R})$ in the usual way, $\langle u_x, \psi\rangle = -\langle u, \psi_x\rangle$, while the derivative with respect to t is the limit of a difference quotient. If equation (1.2) is satisfied in $\mathcal{D}'(\mathbf{R})$ for each t, then we then say that it holds in a weak sense and that u is its weak solution.

We were led to the Burgers equation by our previous work on *peakon* and *shockpeakon* solution of the Degasperis–Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{1.3}$$

an integrable wave equation discovered a few years ago [2, 3]. Indeed, the problems treated in this paper are to some extent "toy problems", but we hope that they might provide some guidance and intuition for the future study of the DP equation.

Equation (1.1) can be obtained formally from the DP equation by substituting $x \mapsto \varepsilon x$, $t \mapsto \varepsilon t$, and then letting $\varepsilon \to 0$. This "high-frequency limit" is a natural thing to try on the DP equation, since it is the same procedure that takes the celebrated integrable Camassa–Holm (CH) shallow water equation [4],

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.4}$$

to the Hunter–Saxton (HS) equation for nematic liquid crystals [5, 6],

$$(u_t + uu_x)_{xx} = u_x u_{xx}. \tag{1.5}$$

The CH and DP equations both admit *peakon* solutions, which are multisoliton solutions of the form

$$u(x,t) = \sum_{k=1}^{n} m_k(t)\, \exp\bigl(-|x - x_k(t)|\bigr), \tag{1.6}$$

where the functions $x_k(t)$ and $m_k(t)$ (positions and momenta of the individual peak-shaped solitons) are required to satisfy a certain system of $2n$ ODEs in order for $u(x,t)$ to satisfy the PDE in a weak sense. In shorthand notation these ODEs are $\dot{x}_k = u(x_k)$, $\dot{m}_k = -(b-1)u_x(x_k)$, where $b = 2$ for the CH equation and $b = 3$ for the DP equation. One can think of this as an integrable mechanical system of n particles on the real line, simililar to, for example, the open Toda lattice. If follows from the rapid decay of $e^{-|x|}$ that $\dot{x}_k = u(x_k) \approx m_k$ when all distances $|x_i - x_j|$ are large, so it agrees with intuition to regard m_k as the momentum of the kth particle. Asympotically (when $t \to \pm\infty$) the particles will spread apart, each moving with its own (nearly) constant velocity which is

nonzero and distinct from the other particles' velocities. The latter is a highly nontrivial fact for the DP equation [7, Theorem 2.4].

There is an analogous class of solutions of the HS equation and the forced Burgers equation (1.2), namely the piecewise linear solutions

$$u(x,t) = \sum_{k=1}^{n} m_k(t) \, |x - x_k(t)| \,. \tag{1.7}$$

In the shorthand notation used above, the governing ODEs take exactly the same form again: $\dot{x}_k = u(x_k)$, $\dot{m}_k = -(b-1) u_x(x_k)$, where $b = 2$ for the HS equation and $b = 3$ for the forced Burgers equation. However, for peakons the term $m_k e^{-|x_k - x_k|}$ usually dominates the other terms in the equation $\dot{x}_k = u(x_k)$, while here we instead have the term $m_k \, |x_k - x_k|$ which is zero while all *other* terms are large. Thus, in contrast to peakons where the interaction is strongly localized, these piecewise linear solitons influence each other more strongly the more separated they are. Although it is a bit hard to develop a useful intuition about these ODEs as a "mechanical" system (perhaps one can think of some kind of expanding gas with long-range correlations), the analogy with peakons still makes it natural to think of the piecewise linear solutions as being composed of some kind of solitons situated at the breakpoints x_k. (But we have not been able to make sense of the idea that the piecewise linear solutions are somehow high-frequency limits of peakons).

In all four cases mentioned above, the ODEs governing the soliton dynamics can be explicitly solved using inverse spectral methods [8, 9, 7, 10, 1, 11]. In the forced Burgers case the ODEs are also easily solved directly by elementary methods, as we will see.

In the Degasperis–Procesi equation (but not in the Camassa–Holm equation) there also appears a more complicated phenomenon, namely discontinuous solutions of the form

$$u(x,t) = \sum_{k=1}^{n} \Big(m_k(t) - s_k(t) \, \operatorname{sgn}\big(x - x_k(t)\big) \Big) \exp\big(- |x - x_k(t)|\big). \tag{1.8}$$

Such *shockpeakons* [12] are governed by $3n$ ODEs for positions x_k, momenta m_k, and shock strengths s_k. Even if one starts with the usual peakon ansatz (1.6), shock solutions of the form (1.8) can form after finite time when a peakon with $m_k > 0$ collides with an *antipeakon* with $m_{k+1} < 0$ moving in the opposite direction. (In the CH equation, such collisions give rise to "zero-strength shocks" where u_x momentarily blows up but u remains continuous, still being of the form (1.6) after the collision [8], and a similar thing occurs for the HS equation [6].) The shockpeakon ODEs have so far only been solved in the trivial case $n = 1$ and in a very particular subcase when $n = 2$. The problem is that the Lax pair for the DP equation, which was crucial for deriving the peakon solution formulas, does not make sense for the weak formulation of the DP equation that is used when working with discontinuous solutions.

The forced Burgers equation (1.2) admits an analogous class of solutions, given by the discontinuous piecewise linear ansatz

$$u(x,t) = \sum_{k=1}^{n} \Bigl(m_k(t)\, |x - x_k(t)| - s_k(t) \operatorname{sgn}\bigl(x - x_k(t)\bigr) \Bigr). \tag{1.9}$$

Such solutions with shocks can form after finite time, even if the initial profile is continuous. Unlike the Degasperis–Procesi case, it turns out here that the extra generality of having jumps in u can be handled without problems.

The outline of the paper is simple: we derive and solve the ODEs governing piecewise linear solutions of the forced Burgers equation (1.2), first in the simpler case (1.7) of continuous solutions (using elementary methods and, for comparison, inverse spectral methods), then in the general case (1.9) of discontinuous solutions (by reduction to the previous case). We conclude with a few examples.

2 Continuous piecewise linear solutions

Theorem 1. *The continuous piecewise linear ansatz* (1.7), $u = \sum m_k |x - x_k|$, *is a weak solution to the linearly forced inviscid Burgers equation* (1.2) *if and only if*

$$\dot{x}_k = \sum_{i=1}^{n} m_i\, |x_k - x_i|, \qquad \dot{m}_k = 2 m_k \sum_{i=1}^{n} m_i \operatorname{sgn}(x_i - x_k), \tag{2.1}$$

for $k = 1, \dots, n$. *For this class of solutions, equation* (1.2) *takes the form*

$$u_t + \frac{1}{2}(u^2)_x = M^2 x - M M_+, \tag{2.2}$$

where $M = \sum_{k=1}^{n} m_k$ *and* $M_+ = \sum_{k=1}^{n} m_k x_k$ *are constants of motion.*

Proof. This is a special case (all $s_k = 0$) of Theorem 7 which is proved later. □

One can assume that all $m_k \neq 0$, since it follows from (2.1) that any vanishing m_k remains identically zero. If we think of x_k and m_k as positions and masses of particles on a line, then the total mass M and the center of mass M_+/M (if $M \neq 0$) are conserved. Note that when $M = 0$ we have the unforced Burgers equation. There are some additional constants of motion $M_2, \dots, M_n$ that come together with $M_1 = M$ from the Lax pair presented in the next section, but we will not need them here [1].

The presence of absolute values and the sign function in (2.1) naturally divides the position space $\mathbf{R}^n$ into sectors. More precisely, to any permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ of the numbers $\{1, 2, \dots, n\}$ one can assign the sector $X_\sigma = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \,|\, x_{\sigma_1} < x_{\sigma_2} < \dots < x_{\sigma_n}\}$. We will concentrate on the sector X_e corresponding to the identity permutation $e = 12 \dots n$, since there is

no loss of generality in assuming that the initial positions $x_k(0)$ are sorted in increasing order:

$$X_e = \{(x_1, x_2, \dots, x_n) \in \mathbf{R}^n \mid x_1 < x_2 < \cdots < x_n\}. \tag{2.3}$$

For positions in X_e the ODEs (2.1) take the form

$$\dot{x}_k = \sum_{i=1}^{n} m_i (x_k - x_i) \operatorname{sgn}(k-i), \qquad \dot{m}_k = 2m_k \sum_{i=1}^{n} m_i \operatorname{sgn}(i-k). \tag{2.4}$$

The following theorem solves this system completely.

Theorem 2. *Given any initial data $\{x_k(0), m_k(0)\}_{k=1}^n$ (with the $x_k(0)$'s ordered or not), the solution of the ODEs* (2.4) *is given by the formulas below, where $M = \sum m_k$ and $M_+ = \sum m_k x_k$ as before, and where the empty sums $\sum_1^0$ and $\sum_{n+1}^n$ in F_0 and F_n are to be interpreted as zero (so that $F_0(t) = e^{-Mt}$ and $F_n(t) = e^{Mt}$).*

- *When $M \neq 0$ the solution of* (2.4) *is*

$$\begin{aligned} x_k(t) &= \frac{M_+}{M} + \frac{e^{Mt}}{M} \left(\sum_{j<k} \big(x_k(0) - x_j(0)\big) m_j(0) \right) \\ &\quad + \frac{e^{-Mt}}{M} \left(\sum_{j>k} \big(x_k(0) - x_j(0)\big) m_j(0) \right), \\ m_k(t) &= \frac{m_k(0)}{F_{k-1}(t) F_k(t)}, \end{aligned} \tag{2.5}$$

  for $k = 1, \dots, n$, where

$$F_k(t) = \frac{e^{Mt}}{M} \left(\sum_{j=1}^{k} m_j(0) \right) + \frac{e^{-Mt}}{M} \left(\sum_{j=k+1}^{n} m_j(0) \right). \tag{2.6}$$

- *When $M = 0$ the solution of* (2.4) *is*

$$\begin{aligned} x_k(t) &= x_k(0) + t \left(\sum_{j<k} \big(x_k(0) - x_j(0)\big) m_j(0) - \sum_{j>k} \big(x_k(0) - x_j(0)\big) m_j(0) \right), \\ m_k(t) &= \frac{m_k(0)}{F_{k-1}(t) F_k(t)}, \end{aligned} \tag{2.7}$$

  for $k = 1, \dots, n$, where

$$F_k(t) = 1 + t \left(\sum_{j=1}^{k} m_j(0) - \sum_{j=k+1}^{n} m_j(0) \right). \tag{2.8}$$

- *Letting $l_k = x_{k+1} - x_k$ for $k = 1, \dots, n-1$, we have in both cases*

$$l_k(t) = l_k(0) F_k(t). \tag{2.9}$$

The proof is presented at the end of this section. As an immediate corollary we obtain information about the original ODEs (2.1).

Theorem 3. *Given initial data $\{x_k(0), m_k(0)\}_{k=1}^n$ to the ODEs* (2.1) *such that $x_1(0) < x_2(0) < \cdots < x_n(0)$ (that is, with the positions in the sector X_e of $\mathbf{R}^n$), the solution is given locally (around $t = 0$) by the formulas of Theorem 2, and this solution is valid as long as the positions $x_k(t)$ remain in X_e.*

A local solution that starts in X_e hits the boundary of X_e whenever $x_k = x_{k+1}$ for at least one k, an event which we refer to as a *collision*. It is clear from (2.9) that a collision occurs when some F_k becomes zero, at which time m_k and m_{k+1} blow up. The local solution is valid up until the time of the first collision. In general a shock will then form, and the continuous ansatz (1.7) will not be able to describe the solution beyond the point of collision. We will return to this in the section about discontinuous solutions.

If all $m_k(0)$'s have the same sign, then (2.6) shows that there are no collisions, so the solution is global. In the case when all are positive, the asymptotic behaviour of this global solution as $t \to +\infty$ is that $x_1 \to M_+/M$ and $m_1 \to M$, while $x_k \to +\infty$ and $m_k \to 0$ for all $k > 1$. When the $m_k(0)$'s have mixed signs, collisions may or may not occur for $t > 0$. For example, in the case $n = 2$ a collision takes place when $F_1(t) = (m_1(0)e^{Mt} + m_2(0)e^{-Mt})/M$ becomes zero, which happens when $m_2(0)/m_1(0) < 0$ and $t = (2M)^{-1} \ln |m_2(0)/m_1(0)|$. Consideration of cases shows that this value of t is positive iff $m_1(0) < 0 < m_2(0)$.

The event when $l_{k-1} = l_k = 0$ is called a triple collision, since three particles come together at one point. The absence of triple collisions in the CH equation is a nontrivial result [8, 13], but for the linearly forced Burgers equation it is much simpler. (Note, however, that triple collisions *are* possible for discontinuous piecewise linear solution; see the examples at the end of the paper.)

Theorem 4. *Collisions occuring in continuous piecewise linear solutions of the linearly forced Burgers equation* (1.2) *cannot be triple collisions.*

Proof. A triple collision would occur if $l_{k-1}(t_0) = 0 = l_k(t_0)$ for some t_0, which amounts to $F_{k-1}(t_0) = 0 = F_k(t_0)$ by (2.9). From the definition of F_k, it is obvious that this is impossible in the case $M = 0$, since we are assuming $m_k \neq 0$. In the case $M \neq 0$, it is also impossible, although less obvious; $F_k(t_0) = 0$ iff $t_0 = \frac{1}{2M} \log \frac{-\sum_{j>k} m_j(0)}{m_k(0) + \sum_{j<k} m_j(0)}$ and the quotient inside the logarithm is positive. So $F_k = F_{k-1} = 0$ iff $\frac{m_k(0)+B}{A} = \frac{B}{m_k(0)+A} < 0$, where $A = \sum_{j<k} m_j(0)$ and $B = \sum_{j>k} m_j(0)$, which requires that $(m_k + A)(m_k + B) = AB$, and hence $m_k(A + m_k + B) = 0$. But this is ruled out by m_k and $A + m_k + B = M$ both being nonzero. $\square$

We finish this section with the postponed proof of the main theorem.

Proof of Theorem 2. Assume to begin with that $x_1(0) < \cdots < x_n(0)$. Then (2.4) is equivalent to (2.1), and we can attack the problem by trying to find $x_k(t)$ and $m_k(t)$ such that the corresponding piecewise linear $u(x,t)$ given by (1.7) satisfies the PDE (2.2). The x_k's divide the real line into $n+1$ intervals which we number by $k = 0,\ldots,n$. In each such interval u takes the form $u(x,t) = a_k(t)x + b_k(t)$. Inserting this into (2.2) yields $\dot{a}_k + a_k^2 = M^2$ and $\dot{b}_k + b_k a_k = -MM_+$, from which (in the case $M \neq 0$)

$$a_k(t) = M\frac{a_k(0)\cosh(Mt) + M\sinh(Mt)}{a_k(0)\sinh(Mt) + M\cosh(Mt)} \tag{2.10}$$

is found immediately, and by making an ansatz for b_k with the same denominator as a_k one also obtains

$$b_k(t) = \frac{a_k(0)M_+\big(1-\cosh(Mt)\big) + M\big(b_k(0) - M_+\sinh(Mt)\big)}{a_k(0)\sinh(Mt) + M\cosh(Mt)}. \tag{2.11}$$

Now $x_k(t)$ and $m_k(t)$ are recovered from the relations $m_k = \frac{1}{2}(a_k - a_{k-1})$ and $x_k = -(b_k - b_{k-1})/(a_k - a_{k-1})$. Because of the algebraic nature of the formulas thus obtained, they satisfy the ODEs (2.4) identically, which shows that the assumption $x_1 < \ldots < x_n$ is immaterial and can be removed. (This will be important later; see the comments after Theorem 8.) The simpler case $M = 0$ (unforced Burgers) is entirely similar, except that

$$a_k(t) = \frac{a_k(0)}{ta_k(0)+1}, \qquad b_k(t) = \frac{b_k(0)}{ta_k(0)+1}. \tag{2.12}$$

(The solution for $M = 0$ can also be obtained by expanding $e^{\pm Mt} = 1 \pm Mt + O(M^2)$ in the solution for $M \neq 0$ and letting $M \to 0$.) $\square$

3 Inverse spectral construction of solutions

The Lax pair

$$-\partial_x^3\phi = zm\phi, \tag{3.1}$$

$$\phi_t = \left[z^{-1}\partial_x^2 + c + u_x - u\partial_x\right]\phi, \tag{3.2}$$

with c an arbitrary constant, is compatible iff $m_t + m_x u + 3mu_x = 0$ and $m_x = u_{xxx}$, under the assumption of sufficient smoothness needed to justify the cross-differentiation. In particular, it is compatible if u evolves according to the derivative Burgers equation (1.1), which can be written as $m_t + m_x u + 3mu_x = 0$ with $m = u_{xx}$. To obtain the linearly forced Burgers equation (1.2) from equation (1.1) the rule $(u^2)_x = 2uu_x$ is used. It is not obvious if all these formal calculations have any relevance to weak solutions, where the smoothness assumptions may be violated. To investigate this, let us say that (3.1) and (3.2)

constitute a *weak Lax pair* if they are satisfied in the weak sense discussed in the introduction (thus ϕ, like u, is a $\mathcal{D}'(\mathbf{R})$-valued function of t, and the equations hold in the space of distributions $\mathcal{D}'(\mathbf{R})$). Solutions u of the form (1.7), $u = \sum m_k |x - x_k|$, do admit a weak Lax pair with $m = u_{xx} = 2\sum_{k=1}^n m_k \delta_{x_k}$, and ϕ is in this case a continuous function (in fact, it is piecewise a quadratic polynomial in x with t-dependent coefficients). The product $m\phi$ in (3.1) is well-defined since the distribution m can be multiplied by the continuous function ϕ. We hope to treat weak Lax pairs in more depth in future papers. Here we just state a theorem which can be verified by careful use of the calculus of distributions.

Theorem 5. *The following are equivalent conditions on a function u of the form* (1.7)*, $u = \sum m_k |x - x_k|$:*

1. *u is a weak solution to the linearly forced Burgers equation* (1.2)*, and $\{x_k, m_k\}$ satisfy equations* (2.1)*.*
2. *u has a weak Lax pair* (3.1)*,* (3.2)*.*

When $u = \sum m_k |x - x_k|$, a solution to equation (3.1) with the asymptotic condition $\phi(x, t; z) = 1$ for $x < x_1(t)$ will be consistent with the time evolution given by (3.2) provided that we choose the constant $c = -M$. Such a solution evaluated at $x > x_n(t)$ will take the form $\phi(x, t; z) = A(t; z)\frac{1}{2}(x - x_n)^2 + B(t; z)(x - x_n) + C(t; z)$, where all three coefficients are polynomials in z, which, by equation (3.2), satisfy $\dot{A} = 0$, $\dot{B} = MB$, and $\dot{C} = \frac{A}{z} + 2MC$ (see [1]). Thus it is consistent with equations (3.1) and (3.2) to impose the condition $A(t; z) = 0$, which together with $\phi = 1$ for $x < x_1$ amounts to the boundary conditions $\phi_x(-\infty) = \phi_{xx}(-\infty) = \phi_{xx}(\infty) = 0$. With these boundary conditions in place, the problem of solving the ODEs (2.1) becomes an isospectral deformation problem which can be solved if one knows how to solve the inverse problem for equation (3.1). This is exactly the inverse problem that was studied in [1] under the additional assumption that all $m_k(0) > 0$. We now give a brief summary of results from that paper.

Theorem 6. *The "Neumann-like discrete cubic string" boundary value problem*

$$-\partial_x^3 \phi = zm\phi, \qquad \phi_x(-\infty) = \phi_{xx}(-\infty) = \phi_{xx}(\infty) = 0,$$

where $m = 2\sum_{k=1}^n m_k \delta_{x_k}$ with all $m_k > 0$, has a spectrum of the form $\{0 = z_0 < z_1 < z_2 < \cdots < z_{n-1}\}$. There is a one-to-one (up to translations of m along the x axis) and onto spectral map $m \mapsto \{M, \mu\}$, where $M = \sum m_k > 0$ and μ is a measure of the form $\mu = \sum_{j=1}^{n-1} b_j \delta_{z_j}$, with $b_j > 0$ for $j = 1, \ldots, n-1$ (see details in [1]). The inverse problem of recovering the discrete measure m from $\{M, \mu\}$ has the explicit solution

$$m_{n-k} = \frac{\mathcal{C}_k \mathcal{D}_k}{2\mathcal{A}_{k+1}\mathcal{A}_k}, \qquad x_{n-k+1} - x_{n-k} \equiv l_{n-k} = -\frac{2\mathcal{A}_k}{\mathcal{D}'_k}. \tag{3.3}$$

in terms of determinants of bimoment matrices constructed out of the measure μ and the constant M (see below).

We recall the following definitions from [1]. Given a measure μ, let

$$\beta_j = \int z^j \, d\mu(z), \qquad I_{ij} = I_{ji} = \iint \frac{z^i \, w^j}{z+w} d\mu(z) d\mu(w). \tag{3.4}$$

Let $\mathcal{A}_0 = \mathcal{B}_0 = \mathcal{C}_0 = \mathcal{D}_0 = 1$, $\mathcal{A}_1 = I_{00} + \frac{1}{2M}$, $\mathcal{D}_1' = \beta_0$, and for other values of k let

$$\mathcal{A}_k = \begin{vmatrix} I_{00} + \frac{1}{2M} & I_{01} & \cdots & I_{0,k-1} \\ I_{10} & I_{11} & \cdots & I_{1,k-1} \\ I_{20} & I_{21} & \cdots & I_{2,k-1} \\ \vdots & \vdots & & \vdots \\ I_{k-1,0} & I_{k-1,1} & \cdots & I_{k-1,k-1} \end{vmatrix},$$

$$\mathcal{B}_k = \begin{vmatrix} I_{00} & I_{01} & \cdots & I_{0,k-1} \\ I_{10} & I_{11} & \cdots & I_{1,k-1} \\ \vdots & \vdots & & \vdots \\ I_{k-1,0} & I_{k-1,1} & \cdots & I_{k-1,k-1} \end{vmatrix}, \qquad \mathcal{C}_k = \begin{vmatrix} I_{11} & I_{12} & \cdots & I_{1k} \\ I_{21} & I_{22} & \cdots & I_{2k} \\ \vdots & \vdots & & \vdots \\ I_{k1} & I_{k2} & \cdots & I_{kk} \end{vmatrix}, \tag{3.5}$$

$$\mathcal{D}_k = \begin{vmatrix} I_{10} & I_{11} & \cdots & I_{1,k-1} \\ I_{20} & I_{21} & \cdots & I_{2,k-1} \\ \vdots & \vdots & & \vdots \\ I_{k0} & I_{k1} & \cdots & I_{k,k-1} \end{vmatrix}, \qquad \mathcal{D}_k' = \begin{vmatrix} \beta_0 & I_{10} & \cdots & I_{1,k-2} \\ \beta_1 & I_{20} & \cdots & I_{2,k-2} \\ \vdots & \vdots & & \vdots \\ \beta_{k-1} & I_{k0} & \cdots & I_{k,k-2} \end{vmatrix}.$$

In all these cases, the index k agrees with the size $k \times k$ of the determinant. Note that $\mathcal{A}_k = \mathcal{B}_k + \frac{1}{2M}\mathcal{C}_{k-1}$ for $k \geq 1$.

Let us analyze the formula (3.3) for l_k in order to compare it with (2.9) obtained earlier. First, (3.2) implies that the linearly forced Burgers equation induces a very simple evolution of the measure μ, namely $\mu(z;t) = e^{Mt}\mu(z;0)$. Because of this it is easy to factor out the time dependence from all the determinants involved in (3.3). This elementary exercise leads to $l_k(t) = l_k(0)F_k(t)$, where

$$F_k(t) = \frac{\mathcal{B}_{n-k}(0)\, e^{Mt} + \frac{1}{2M} C_{n-k-1}(0)\, e^{-Mt}}{\mathcal{B}_{n-k}(0) + \frac{1}{2M} C_{n-k-1}(0)}. \tag{3.6}$$

This is in full agreement with (2.6) and (2.9). The formula for m_k can be checked in a similar way.

4 Discontinuous piecewise linear solutions

Theorem 7. *The discontinuous piecewise linear ansatz* (1.9), $u = \sum (m_k \, |x - x_k| - s_k \operatorname{sgn}(x - x_k))$, *is a weak solution of the linearly forced inviscid Burgers equation*

(1.2) *if and only if*

$$\begin{aligned}
\dot{x}_k &= \sum_{i=1}^{n} \big(m_i \, |x_k - x_i| + s_i \operatorname{sgn}(x_i - x_k) \big), \\
\dot{m}_k &= 2 m_k \sum_{i=1}^{n} m_i \operatorname{sgn}(x_i - x_k), \qquad \dot{s}_k = s_k \sum_{i=1}^{n} m_i \operatorname{sgn}(x_i - x_k),
\end{aligned} \tag{4.1}$$

for $k = 1, \dots, n$*. For this class of solutions, equation* (1.2) *takes the form*

$$u_t + \frac{1}{2}(u^2)_x = M^2 x - M(M_+ + S), \tag{4.2}$$

with $M = \sum m_k$ *and* $M_+ = \sum m_k x_k$ *as before, and with* $S = \sum s_k$*. The quantities* M *and* $M_+ + S$ *are constants of motion, and so is* s_k^2/m_k *for* $k = 1, \dots, n$ *(provided that* $m_k \neq 0$*).*

Proof. We will repeatedly use the following distributional formula valid for an arbitrary piecewise differentiable function f with points of discontinuity at $x_1, x_2, \dots, x_n$: $f_x = \{f_x\} + \sum_{k=1}^{n} [f]_k \delta_{x_k}$, where $\{f_x\}$ means the ordinary derivative taken away from discontinuities and $[f]_k = f(x_k^+) - f(x_k^-)$ denotes the jump at x_k. Moreover, $u_t = \sum_k \big(\dot{m}_k |x - x_k| - (m_k \dot{x}_k + \dot{s}_k) \operatorname{sgn}(x - x_k) + 2 s_k \dot{x}_k \delta_{x_k} \big)$. Now the left-hand side of (1.2), $u_t + \frac{1}{2}(u^2)_x$, must be a function since the right-hand side is a function; hence all Dirac deltas must cancel out. Similarly, there must be no Dirac deltas in the first or second x derivatives of $u_t + \frac{1}{2}(u^2)_x$, These conditions give, in turn,

$$\begin{aligned}
&0 = 2 s_k \dot{x}_k + \tfrac{1}{2}[u^2]_k, \qquad 0 = -2(m_k \dot{x}_k + \dot{s}_k) + \tfrac{1}{2}[\{(u^2)_x\}]_k, \\
&0 = 2 \dot{m}_k + \tfrac{1}{2}[\{(u^2)_{xx}\}]_k.
\end{aligned} \tag{4.3}$$

An elementary computation of jumps for the case of piecewise continuous functions now produces (4.1). The coefficients of the forcing term in the PDE are identified from the smooth part of the term $\frac{1}{2}(u^2)_x$, while the constants of motion follow from (4.1). □

Weak solutions to an initial value problem are usually not unique unless the PDE is supplemented with a so-called entropy condition that picks out the "physical" solution. In the case of the Burgers equation this condition requires u to jump down, not up, at discontinuities. This is satisfied by the ansatz (1.9) if all shock strenght s_k are nonnegative, so we will assume $s_k \geq 0$ from now on.

When considering the initial value problem for the ODEs (4.1) we can assume without loss of generality that $x_1(0) < x_2(0) < \dots < x_n(0)$. Thus on a sufficiently small time interval we will still have $x_1(t) < x_2(t) < \dots < x_n(t)$; in other words, the positions stay in the sector X_e (see (2.3)). In X_e the equations

(4.1) can be written as

$$\begin{aligned} \dot{x}_k &= \sum_{i=1}^{n} \bigl(m_i \operatorname{sgn}(k-i)(x_k - x_i) - s_i \operatorname{sgn}(k-i) \bigr), \\ \dot{m}_k &= 2m_k \sum_{i=1}^{n} m_i \operatorname{sgn}(i-k), \qquad \dot{s}_k = s_k \sum_{i=1}^{n} m_i \operatorname{sgn}(i-k), \end{aligned} \tag{4.4}$$

for $k = 1, \dots, n$. These equations can be solved explicitly, since the simple change of variables in the following theorem reduces them to the ODEs already solved in Theorem 2.

Theorem 8. *If $\{x_k, m_k, s_k\}_{k=1}^{n}$ satisfy (4.4), if all $m_k(0) \neq 0$, and if*

$$y_k = x_k + s_k/m_k, \tag{4.5}$$

then $\{y_k, m_k\}_{k=1}^{n}$ satisfy (2.4) (with y_k taking the place of x_k everywhere).

Proof. Straightforward calculation. □

Note that the initital values $y_k(0)$ will not necessarily be distinct or sorted in increasing order even though the $x_k(0)$'s are, but this does not matter since the solution formulas of Theorem 2 are valid for any initial conditions. So Theorem 2 gives us $y_k(t)$ and $m_k(t)$ (note that $\sum m_k y_k = \sum (m_k x_k + s_k) = M_+ + S$ replaces M_+ in the solution formula (2.5)), and we can then recover $s_k(t)$ from the fact that s_k^2/m_k is constant for each k; this gives $s_k(t) = s_k(0)/\sqrt{F_{k-1}(t)F_k(t)}$, and allows us to also recover $x_k(t) = y_k(t) - s_k(t)/m_k(t)$. This solution $\{x_k, m_k, s_k\}$ to (4.4) is also the solution to (4.1), at least locally in some time interval around $t = 0$ (so that the x_k's remain in the sector X_e).

For illustration, here is the general solution with shocks in the case $n = 2$, when $M \neq 0$, $m_1(0) \neq 0$, $m_2(0) \neq 0$:

$$\begin{aligned} m_1(t) &= \frac{m_1(0)}{F_0(t)F_1(t)}, \qquad s_1(t) = \frac{s_1(0)}{\sqrt{F_0(t)F_1(t)}}, \\ m_2(t) &= \frac{m_2(0)}{F_1(t)F_2(t)}, \qquad s_2(t) = \frac{s_2(0)}{\sqrt{F_1(t)F_2(t)}}, \\ x_1(t) &= \frac{M_+ + S - K m_2(0)\, e^{-Mt}}{M} - \frac{s_1(0)}{m_1(0)} \sqrt{F_0(t)F_1(t)}, \\ x_2(t) &= \frac{M_+ + S + K m_1(0)\, e^{Mt}}{M} - \frac{s_2(0)}{m_2(0)} \sqrt{F_1(t)F_2(t)}, \\ F_0(t) &= e^{-Mt}, \qquad F_1(t) = \frac{m_1(0)\, e^{Mt} + m_2(0)\, e^{-Mt}}{M}, \qquad F_2(t) = e^{Mt}, \\ K &= x_2(0) - x_1(0) + \frac{s_2(0)}{m_2(0)} - \frac{s_1(0)}{m_1(0)}. \end{aligned} \tag{4.6}$$

In the continuous case (2.1) we assumed all $m_k \neq 0$, but for (4.1) it does make sense to have $m_k = 0$ provided that the corresponding s_k is nonzero. If

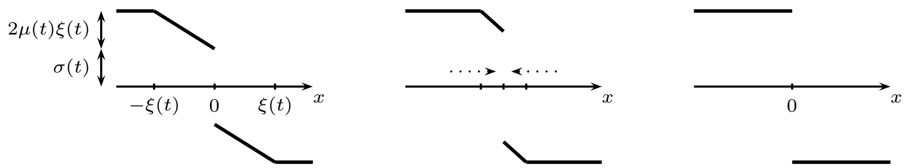

Figure 1: Left/middle: Wave profile $u(x,t)$ as given by (4.7) at two different times $t < t_{\mathrm{coll}}$, with $\xi(t)$ decreasing towards zero at a constant rate. Right: Stationary profile after collision ($t \geq t_{\mathrm{coll}}$).

$m_k(0) = 0$, then clearly $m_k(t) = 0$ for all t, and the above solution procedure does not work. But this is easily fixed: just write down the general solution obtained for a nonzero initial value $m_k(0) = a$, and let $a \to 0$ there.

We will finish with a few examples that show how to deal with the solution when it hits the boundary of the sector X_e.

Example. A particular antisymmetric solution of (4.1) with $n = 3$ is given by $-x_1 = x_3 \equiv \xi > 0$, $x_2 = 0$, $-m_1 = m_3 \equiv \mu > 0$, $m_2 = 0$, $s_1 = s_3 = 0$, $s_2 \equiv \sigma \geq 0$, where $\xi(t) = \xi(0)F(t) - \sigma(0)t$, $\mu(t) = \mu(0)/F(t)$, $\sigma(t) = \sigma(0)/F(t)$, with $F(t) = 1 - 2\mu(0)t$. (These formulas are obtained either by reducing (4.1) to ODEs for ξ, μ, σ and solving them directly; or by assuming $m_2(0) = a \neq 0$, changing variables to $y_1 = x_1$, $y_2 = x_2 + s_2/m_2$, $y_3 = x_3$, writing down the general solution using Theorems 8 and 2, and letting $a \to 0$; or simply by noting that $M = 0$ so that we are dealing with the unforced Burgers equation whose solution can be found in the textbook way using characteristics.) Since $M_+ + S = 2\mu\xi + \sigma$ is constant in time, the wave profile (see Figure 1) is

$$
\begin{aligned}
u(x,t) &= -\mu(t)\,|x+\xi(t)| + \mu(t)\,|x-\xi(t)| - \sigma(t)\,\mathrm{sgn}(x) \\
&= \begin{cases} 2\mu(0)\xi(0) + \sigma(0), & x < -\xi(t), \\ -2\mu(t)x + \sigma(t), & -\xi(t) \leq x < 0, \\ 0, & x = 0, \\ -2\mu(t)x - \sigma(t), & 0 < x \leq \xi(t), \\ -\big(2\mu(0)\xi(0) + \sigma(0)\big), & \xi(t) < x. \end{cases}
\end{aligned}
\tag{4.7}
$$

If $\sigma(0) = 0$ then this is a shockless solution (with $n = 2$ really, since there is neither mass nor shock at the site $x_2 = 0$). It is defined until $\xi(t) = \xi(0)F(t)$ becomes zero at time $t_{\mathrm{coll}} = \big(2\mu(0)\big)^{-1}$. Then x_1 and x_3 collide at $x = 0$ while m_1 and m_3 blow up to $-\infty$ and $+\infty$, respectively. However, u remains bounded, and tends to a shock profile: $u(x,t) \to -2\mu(0)\xi(0)\,\mathrm{sgn}(x)$ as $t \nearrow t_{\mathrm{coll}}$. This illustrates that shocks can form naturally even if they are not present in the initial wave profile. The profile will be stationary after the collision, because its continued evolution is given by the $n = 1$ case of (4.1) ($\dot{x}_1 = m_1$, $\dot{m}_1 = \dot{s}_1 = 0$) with $x_1 = 0$, $m_1 = 0$, $s_1 = 2\mu(0)\xi(0)$. Consequently, $u(x,t) = -2\mu(0)\xi(0)\,\mathrm{sgn}(x)$ for all $t \geq t_{\mathrm{coll}}$.

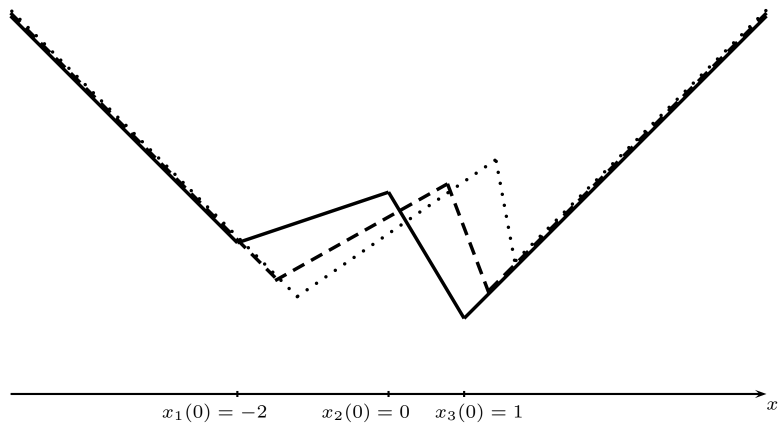

Figure 2: Solid: Continuous initial wave profile $u(x, 0)$. Dashed/dotted: $u(x, t)$ at times $t = \ln \frac{4}{3}$ and $t = \ln \frac{5}{3}$, respectively.

If $\sigma(0) > 0$ there is a shock waiting at the origin between the two approaching particles (as in Figure 1). The solution hits the boundary of the sector X_e when $\xi(t)$ becomes zero at time $t_{\mathrm{coll}} = \big(2\mu(0) + \sigma(0)/\xi(0)\big)^{-1}$. Then $x_1 = x_2 = x_3 = 0$, which illustrates that triple collisions may occur when shocks are present. Since the collision occurs earlier than in the shockless case, $F(t)$ has not yet reached zero at the time of collision; hence m_1 and m_3 do not blow up in this case. Again, u tends to a stationary shock profile: $u(x,t) = -\big(2\mu(0)\xi(0) + \sigma(0)\big)\operatorname{sgn}(x)$ for all $t \geq t_{\mathrm{coll}}$.

Example. Consider now the shockless ODEs (2.1) with $n = 3$ and initial data $m_1(0) = \frac{2}{3}$, $m_2(0) = -1$ and $m_3(0) = \frac{4}{3}$, so that $M = 1$. We assume $x_1(0) < x_2(0) < x_3(0)$ but leave them otherwise unspecified. Since $u = \pm(Mx - M_+)$ as $x \to \pm\infty$, and since the slope u_x jumps by $2m_k$ at each x_k, the initial profile $u(x, 0)$ consists of line segments with slope -1, $\frac{1}{3}$, $-\frac{5}{3}$ and 1, joined at the points $(x_k, u(x_k, 0))$. Figure 2 illustrates this for the particular values $x_1(0) = -2$, $x_2(0) = 0$, $x_3(0) = 1$. Note that if the lines $u = \pm(Mx - M_+)$ to the left and to the right are continued, they intersect on the x axis at the center of mass $x = M_+/M$ ($= 0$ in the figure), which is a constant of motion.

Recall that $l_1 = x_2 - x_1$ and $l_2 = x_3 - x_2$. From (2.6) we obtain $F_0(t) = e^{-t}$, $F_1(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-t}$, $F_2(t) = -\frac{1}{3}e^t + \frac{4}{3}e^{-t}$, and $F_3(t) = e^t$. Equations (2.5) and (2.9) give $x_1(t) = x_1(0) + (1 - e^{-t})\big(\frac{1}{3}l_1(0) + \frac{4}{3}l_2(0)\big)$, $x_2(t) = x_1(t) + l_1(0)F_1(t)$, and $x_3(t) = x_2(t) + l_2(0)F_2(t)$. There is a collision between x_2 and x_3 when $F_2(t)$ becomes zero, which happens at time $t = t_{\mathrm{coll}} = \ln 2$ when $e^t = 2$. At that time we have $F_0 = \frac{1}{2}$, $F_1 = \frac{3}{2}$, $F_2 = 0$, $F_3 = 2$, hence by (2.5) $m_1 = m_1(0)/F_0F_1 = \frac{8}{9}$,

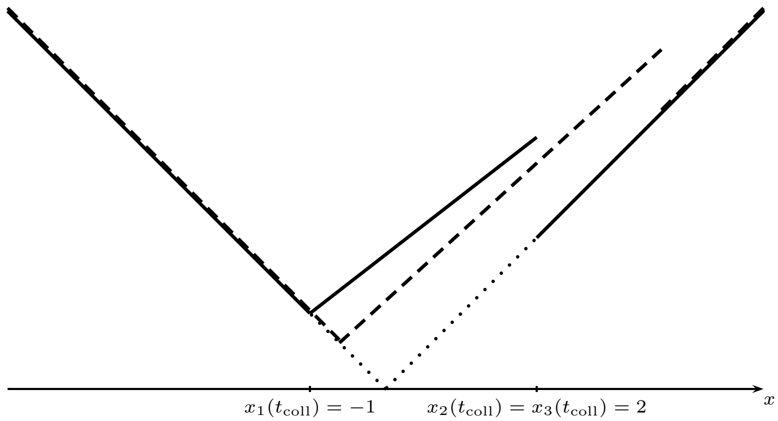

Figure 3: Solid: Discontinuous wave profile $u(x,t)$ formed at the instant of collision $t = t_{\text{coll}} = \ln 2$. Dashed: $u(x,t)$ at time $t = t_{\text{coll}} + \frac{1}{2}$. Dotted: No more collisions occur, and $u(x,t) \to |x|$ as $t \to +\infty$.

$m_2 = -\infty$, $m_3 = +\infty$. As for the wave profile u, we have

$$\begin{aligned} u(x_1(t),t) &= m_2 l_1 + m_3(l_1 + l_2) = (M - m_1)\, l_1 + m_3 l_2 \\ &= (F_1 - m_1(0)/F_0)\, l_1(0) + m_3(0) l_2(0)/F_3 \\ &\to \tfrac{1}{6}\, l_1(0) + \tfrac{2}{3}\, l_2(0), \qquad \text{as } t \nearrow t_{\text{coll}}, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} u(x_2(t),t) - u(x_3(t),t) &= (m_1 l_1 + m_3 l_2) - (m_1(l_1 + l_2) + m_2 l_2) \\ &= (m_3 - m_1 - m_2)\, l_2 \\ &= \big(m_3(0)/F_3 - m_1(0) F_2/F_0 F_1 - m_2(0)/F_1\big)\, l_2(0) \\ &\to \tfrac{4}{3}\, l_2(0), \qquad \text{as } t \nearrow t_{\text{coll}}. \end{aligned} \tag{4.9}$$

Thus the limiting wave profile at $t = t_{\text{coll}}$ consists of a line segment with slope -1, joined to a line segment with slope $-1 + 2 \cdot \frac{8}{9} = \frac{7}{9}$ at $x = x_1(t_{\text{coll}}) = x_1(0) + \frac{1}{2}\big(\frac{1}{3} l_1(0) + \frac{4}{3} l_2(0)\big)$ and height $u = \frac{1}{6}\, l_1(0) + \frac{2}{3}\, l_2(0)$; the profile jumps down by $\frac{4}{3} l_2(0)$ at $x = x_2(t_{\text{coll}}) = x_3(t_{\text{coll}}) = x_1(t_{\text{coll}}) + \frac{3}{2} l_1(0)$, and continues from there with slope 1. See Figure 3.

The continued evolution of the profile for $t \geq t_{\text{coll}}$ is illustrated in Figure 3; it is given by the shock ODEs (4.1) with $n = 2$, using a new set of variables whose initial values at $t = t_{\text{coll}}$ are $\tilde{x}_1 = x_1(t_{\text{coll}})$, $\tilde{x}_2 = x_2(t_{\text{coll}})$, $\tilde{m}_1 = \frac{8}{9}$, $\tilde{m}_2 = \frac{1}{9}$, $\tilde{s}_1 = 0$, and $\tilde{s}_2 = \frac{2}{3} l_2(0)$. In terms of the new time variable $\tau = t - t_{\text{coll}} \geq 0$ one finds from the general solution (4.6) that, for example,

$$\tilde{x}_2(\tau) - \tilde{x}_1(\tau) = \left(\tilde{x}_2(0) - \tilde{x}_1(0) + \frac{\tilde{s}_2(0)}{\tilde{m}_2(0)}\right) \tilde{F}_1(\tau) - \frac{\tilde{s}_2(0)}{\tilde{m}_2(0)} \sqrt{\tilde{F}_1(\tau) \tilde{F}_2(\tau)}, \tag{4.10}$$

where $\tilde{F}_1(\tau) = \frac{8}{9}e^{\tau} + \frac{1}{9}e^{-\tau}$ and $\tilde{F}_2(\tau) = e^{\tau}$. Writing this expression as $\tilde{x}_2 - \tilde{x}_1 = (A+B)\tilde{F}_1 - B\sqrt{\tilde{F}_1\tilde{F}_2}$, we see that it is zero if $F_1(\tau) = 0$, which can never happen, or if $(A+B)^2F_1 = B^2F_2$, which is the same as $e^{-2\tau} = 9((A+B)/B)^2 - 8$ that can't happen either since the right-hand side is > 1 and the left-hand side is ≤ 1 for $\tau \geq 0$. The conclusion is that, in this example, $\tilde{x}_2(\tau) - \tilde{x}_1(\tau)$ remains positive for all $\tau > 0$, so there are no more collisions. Instead, as τ (or t) $\to +\infty$, we have $\tilde{x}_1 \to 0$, $\tilde{x}_2 \to +\infty$, $\tilde{m}_1 \to M$, $\tilde{m}_2 \to 0$, and $\tilde{s}_2 \to 0$. Thus, $u(x,t)$ approaches the limiting wave profile $u(x, +\infty) = |x|$.

Acknowledgments. *HL's participation in NEEDS 2007 was supported by a travel grant from the Knut and Alice Wallenberg Foundation. JS is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC), Grant No. 138591-04.*

References

[1] Jennifer Kohlenberg, Hans Lundmark, and Jacek Szmigielski. The inverse spectral problem for the discrete cubic string. *Inverse Problems*, 23:99–121, 2007.

[2] A. Degasperis and M. Procesi. Asymptotic integrability. In A. Degasperis and G. Gaeta, editors, *Symmetry and perturbation theory (Rome, 1998)*, pages 23–37. World Scientific Publishing, River Edge, NJ, 1999.

[3] A. Degasperis, D. D. Holm, and A. N. W. Hone. A new integrable equation with peakon solutions. *Theoretical and Mathematical Physics*, 133:1463–1474, 2002.

[4] Roberto Camassa and Darryl D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.

[5] John K. Hunter and Ralph Saxton. Dynamics of director fields. *SIAM J. Appl. Math.*, 51(6):1498–1521, 1991.

[6] John K. Hunter and Yu Xi Zheng. On a completely integrable nonlinear hyperbolic variational equation. *Phys. D*, 79(2-4):361–386, 1994.

[7] Hans Lundmark and Jacek Szmigielski. Degasperis–Procesi peakons and the discrete cubic string. *IMRP Int. Math. Res. Pap.*, 2005(2):53–116, 2005.

[8] R. Beals, D. Sattinger, and J. Szmigielski. Multipeakons and the classical moment problem. *Advances in Mathematics*, 154:229–257, 2000.

[9] Hans Lundmark and Jacek Szmigielski. Multi-peakon solutions of the Degasperis–Procesi equation. *Inverse Problems*, 19:1241–1245, December 2003.

[10] Richard Beals, David H. Sattinger, and Jacek Szmigielski. Inverse scattering solutions of the Hunter–Saxton equation. *Appl. Anal.*, 78(3-4):255–269, 2001.

[11] Alberto Bressan and Adrian Constantin. Global conservative solutions of the Camassa–Holm equation. *Arch. Ration. Mech. Anal.*, 183(2):215–239, 2007.

[12] H. Lundmark. Formation and dynamics of shock waves in the Degasperis–Procesi equation. *J. Nonlinear Sci.*, 17(3):169–198, 2007.

[13] Helge Holden and Xavier Raynaud. Global conservative multipeakon solutions of the Camassa–Holm equation. *J. Hyperbolic Differ. Equ.*, 4(1):39–64, 2007.